Partial Differential Equations -Resit exam

You have 3 hours to answer the following questions, and 20 minutes of scan time. Everyone receives 10 points for turning in their exam. This exam is open book and open note, but not open internet. Justify all work with appropriate theorems. Late answers will not be accepted.

(1) (20 points total) (8 points) a) Classify the following partial differential equation

$$u_{xx} + 2u_{xy} + 17u_{yy} = 0 \quad (x, y) \in \mathbb{R}^2$$

and put it in canonical form with the variables $(\eta, \xi) \in \mathbb{R}^2$
(5 points) b) Where does the solution $u(\eta, \xi)$ attain its maximum if (η, ξ) are restricted to the unit disk $(\eta^2 + \xi^2 \leq 1)$ and why?
(7 points) c) Find the solution $u(x, y)$ such that $u(\eta, \xi) = h(\eta)$ on the boundary of the unit disk $\eta^2 + \xi^2 = 1$. You may leave your answer in terms of an integral involving x and y.

(2) (15 points) a) Solve for $u(x, t)$ satisfying

$$\begin{aligned} &u_{tt} = c^2 u_{xx} + k \quad x \in (0, l),\ t \in \mathbb{R} \\ &u(0, t) = u_x(l, t) = 0 \\ &u_t(x, 0) = V \end{aligned}$$

here k and V are constants, using the following steps:
i) Construct a polynomial $v = Ax^2 + Bx + C$ with A, B, C appropriately chosen constants which solves the boundary conditions and the PDE.
ii) Use the principle of superposition to find a new equation which w satisfies, where $u = v + w$.
iii) Solve the problem for w using separation of variables and use part ii) to find u.
(15 points) b) Solve for $u(x, t)$ satsifying

$$\begin{aligned} &u_{tt} = c^2 u_{xx} \quad -\infty < x, t < \infty \\ &u(0, t) = 0 \\ &u_t(x, 0) = V e^{-|x|} \end{aligned}$$

using the Fourier transform in x. You may leave your solution in terms of an inverse Fourier transform.

(3) (20 points) Solve the diffusion equation

$$u_t - k u_{xx} + bt^2 u = 0 \quad -\infty < x < \infty \quad u(x, 0) = \phi(x)$$

(Hint, find the solutions of the ODE $w_t + bt^2 w = 0$. Then make the change of variables $u(x, t) = w(t)v(x, t)$ and find an equation for $v(x, t)$.)

(4) (20 points) Find the one dimensional Green's function for the Laplacian on the interval $(0, l)$. It must have the following properties:
i) It solves $G''(x) = 0$ for $x \neq 0$
ii) $G(0) = G(l) = 0$
iii) $G(x)$ is continuous at x_0 and $G(x) + \frac{1}{2}|x - x_0|$ is harmonic at x_0.
Verify that your Green's function has all three of these properties.